

# Frequent-Itemset Mining using Locality-Sensitive Hashing

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**Abstract.** The Apriori algorithm is a classical algorithm for the frequent itemset mining problem. A significant bottleneck in Apriori is the number of I/O operation involved, and the number of candidates it generates. We investigate the role of LSH techniques to overcome these problems, without adding much computational overhead. We propose randomized variations of Apriori that are based on asymmetric LSH defined over Hamming distance and Jaccard similarity.

## 1 Introduction

Mining *frequent itemsets* in a transactions database appeared first in the context of analyzing supermarket transaction data for discovering association rules [2, 1], however this problem has, since then, found applications in diverse domains like finding correlations [12], finding episodes [8], clustering [13]. Mathematically, each transaction can be regarded as a subset of the items (“itemset”) those that present in the transaction. Given a database  $\mathcal{D}$  of such transactions and a support threshold  $\theta \in (0, 1)$ , the primary objective of frequent itemset mining is to identify  $\theta$ -frequent itemsets (denoted by FI, these are subsets of items that appear in at least  $\theta$ -fraction of transactions).

Computing FI is a challenging problem of data mining. The question of deciding if there exists any FI with  $k$  items is known to be NP-complete [6] (by relating it to the existence of bi-cliques of size  $k$  in a given bipartite graph) but on a more practical note, simply checking support of any itemset requires reading the transaction database – something that is computationally expensive since they are usually of an extremely large size. The state-of-the-art approaches try to reduce the number of candidates, or not generate candidates at all. The best known approach in the former line of work is the celebrated Apriori algorithm [2].

Apriori is based on the *anti-monotonicity property* of partially-ordered sets which says that no superset of an infrequent itemset can be frequent. This algorithm works in a bottom-up fashion by generating itemsets of size  $l$  in level  $l$ , starting at the first level. After finding frequent itemsets at level  $l$  they are joined pairwise to generate  $l+1$ -sized *candidate itemsets*; FI are identified among the candidates by computing their support explicitly from the data. The algorithm terminates when no more candidates are generated. Broadly, there are

two downsides to this simple but effective algorithm. The first one is that the algorithm has to compute support <sup>3</sup> of every itemset in the candidate, even the ones that are highly infrequent. Secondly, if an itemset is infrequent, but all its subsets are frequent, Apriori doesn't have any easy way of detecting this without reading every transaction of the candidates.

A natural place to look for fast algorithms over large data are randomized techniques; so we investigated if LSH could be of any help. An earlier work by Cohen *et al.* [4] was also motivated by the same idea but worked on a different problem (see Section 1.2). LSH is explained in Section 2, but roughly, it is a randomized hashing technique which allows efficient retrieval of approximately "similar" elements (here, itemsets).

### 1.1 Our contribution

In this work, we propose LSH-Apriori – a basket of three explicit variations of Apriori that uses LSH for computing FI. LSH-Apriori handles both the above mentioned drawbacks of the Apriori algorithm. First, LSH-Apriori significantly cuts down on the number of infrequent candidates that are generated, and further due to its dimensionality reduction property saves on reading every transaction; secondly, LSH-Apriori could efficiently filter out those infrequent itemset without looking every candidate. The first two variations essentially reduce computing FI to the approximate nearest neighbor (cNN) problem for Hamming distance and Jaccard similarity. Both these approaches can drastically reduce the number of false candidates without much overhead, but has a non-zero probability of error in the sense that some frequent itemset could be missed by the algorithm. Then we present a third variation which also maps FI to elements in the Hamming space but avoids the problem of these false negatives incurring a little cost of time and space complexity. Our techniques are based on asymmetric LSH [11] and LSH with one-sided error [9] which are proposed very recently.

### 1.2 Related work

There are a few hash based heuristic to compute FI which outperform the Apriori algorithm and PCY [10] is one of the most notable among them. PCY focuses on using hashing to efficiently utilize the main memory over each pass of the database. However, our objective and approach both are fundamentally different from that of PCY.

The work that comes closest to our work is by Cohen *et al.* [4]. They developed a family of algorithms for finding interesting associations in a transaction database, also using LSH techniques. However, they specifically wanted to avoid any kind of filtering of itemsets based on itemset support. On the other hand, our problem is the vanilla frequent itemset mining which requires filtering itemsets satisfying a given minimum support threshold.

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<sup>3</sup> Note that computing support is an I/O intensive operation and involves reading every transaction.

### 1.3 Organization of the paper

In Section 2, we introduce the relevant concepts and give an overview of the problem. In Section 3, we build up the concept of LSH-Apriori which is required to develop our algorithms. In Section 4, we present three specific variations of LSH-Apriori for computing FI. Algorithms of Subsections 4.1 and 4.2 are based on Hamming LSH and Minhashing, respectively. In Subsection 4.3, we present another approach based on CoveringLSH which overcomes the problem of producing false negatives. In Section 5, we summarize the whole discussion.

## 2 Background

Notations			
$\mathcal{D}$	Database of transactions: $\{t_1, \dots, t_n\}$	$n$	Number of transactions
$\mathcal{D}_l$	FI of level- $l$ : $\{I_1, \dots, I_{m_l}\}$	$\theta$	Support threshold, $\theta \in (0, 1)$
$\alpha_l$	Maximum support of any item in $\mathcal{D}_l$	$m$	Number of items
$\varepsilon$	Error tolerance in LSH, $\varepsilon \in (0, 1)$	$m_l$	Number of FI of size $l$
$\delta$	Probability of error in LSH, $\delta \in (0, 1)$	$ v $	Number of 1's in $v$

The input to the classical frequent itemset mining problem is a database  $\mathcal{D}$  of  $n$  transactions  $\{T_1, \dots, T_n\}$  over  $m$  items  $\{i_1, \dots, i_m\}$  and a support threshold  $\theta \in (0, 1)$ . Each transaction, in turn, is a subset of those items. Support of itemset  $I \subseteq \{i_1, \dots, i_m\}$  is the number of transactions that contain  $I$ . The objective of the problem is to determine *every itemset* with support at least  $\theta n$ . We will often identify an itemset  $I$  with its transaction vector  $\langle I[1], I[2], \dots, I[n] \rangle$  where  $I[j]$  is 1 if  $I$  is contained in  $T_j$  and 0 otherwise. An equivalent way to formulate the objective is to find itemsets with at least  $\theta n$  1's in their transaction vectors. It will be useful to view  $\mathcal{D}$  as a set of  $m$  transaction vectors, one for every item.

### 2.1 Locality Sensitive Hashing

We first briefly explain the concept of locality sensitive hashing (LSH).

**Definition 1 (Locality sensitive hashing [7]).** *Let  $S$  be a set of  $m$  vectors in  $\mathbb{R}^n$ , and  $U$  be the hashing universe. Then, a family  $\mathcal{H}$  of functions from  $S$  to  $U$  is called as  $(S_0, (1 - \varepsilon)S_0, p_1, p_2)$ -sensitive (with  $\varepsilon \in (0, 1]$  and  $p_1 > p_2$ ) for the similarity measure  $\text{Sim}(\cdot, \cdot)$  if for any  $x, y \in S$ :*

- if  $\text{Sim}(x, y) \geq S_0$ , then  $\Pr_{h \in \mathcal{H}} [h(x) = h(y)] \geq p_1$ ,
- if  $\text{Sim}(x, y) \leq (1 - \varepsilon)S_0$ , then  $\Pr_{h \in \mathcal{H}} [h(x) = h(y)] \leq p_2$ .

Not all similarity measures have a corresponding LSH. However, the following well-known result gives a sufficient condition for existence of LSH for any  $\text{Sim}$ .

**Lemma 1** *If  $\Phi$  is a strict monotonic function and a family of hash function  $\mathcal{H}$  satisfies  $\Pr_{h \in \mathcal{H}} [h(x) = h(y)] = \Phi(\text{Sim}(x, y))$  for some  $\text{Sim} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \{0, 1\}$ , then the conditions of Definition 1 are true for  $\text{Sim}$  for any  $\varepsilon \in (0, 1)$ .*

The similarity measures that are of our interest are *Hamming* and *Jaccard* over binary vectors. Let  $|x|$  denote the Hamming weight of a binary vector  $x$ . Then, for vectors  $x$  and  $y$  of length  $n$ , Hamming distance is defined as  $\text{Ham}(x, y) = |x \oplus y|$ , where  $x \oplus y$  denotes a vector that is element-wise Boolean XOR of  $x$  and  $y$ . Jaccard similarity is defined as  $\langle x, y \rangle / |x \vee y|$ , where  $\langle x, y \rangle$  indicates inner product, and  $x \vee y$  indicates element-wise Boolean OR of  $x$  and  $y$ . LSH for these similarity measures are simple and well-known [7, 5, 3]. We recall them below; here  $I$  is some subset of  $\{1, \dots, n\}$  (or,  $n$ -length transaction vector).

**Definition 2 (Hash function for Hamming distance).** *For any particular bit position  $i$ , we define the function  $h_i(I) := I[i]$ . We will use hash functions of the form  $g_J(I) = \langle h_{j_1}(I), h_{j_2}(I), \dots, h_{j_k}(I) \rangle$ , where  $J = \{j_1, \dots, j_k\}$  is a subset of  $\{1, \dots, n\}$  and the hash values are binary vectors of length  $k$ .*

**Definition 3 (Minwise Hash function for Jaccard similarity).** *Let  $\pi$  be some permutations over  $\{1, \dots, n\}$ . Treating  $I$  as a subset of indices, we will use hash functions of the form  $h_\pi(I) = \arg \min_i \pi(i)$  for  $i \in I$ .*

The probabilities that two itemsets hash to the same value for these hash functions are related to their Hamming distance and Jaccard similarity, respectively.

## 2.2 Apriori algorithm for frequent itemset mining

As explained earlier, Apriori works in level-by-level, where the objective of level- $l$  is to generate all  $\theta$ -frequent itemsets with  $l$ -items each; for example, in the first level, the algorithm simply computes support of individual items and retains the ones with support at least  $\theta n$ . Apriori processes each level, say level- $(l + 1)$ , by joining all pairs of  $\theta$ -frequent *compatible itemsets* generated in level- $l$ , and further filtering out the ones which have support less than  $\theta n$  (support computation involves fetching the actual transactions from disk). Here, two candidate itemsets (of size  $l$  each) are said to be compatible if their union has size exactly  $l + 1$ . A high-level pseudocode of Apriori is given in Algorithm 1.

**Input:** Transaction database  $\mathcal{D}$ , support threshold  $\theta$ ;  
**Result:**  $\theta$ -frequent itemsets;

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1  $l = 1$  /* level */;
2  $F = \{\{x\} \mid \{x\} \text{ is } \theta\text{-frequent in } \mathcal{D}\}$  /* frequent itemsets in level-1 */;
3 Output  $F$ ;
4 while  $F$  is not empty do
5    $l = l + 1$ ;
6    $C = \{I_a \cup I_b \mid I_a \in F, I_b \in F, I_a \text{ and } I_b \text{ are compatible}\}$ ;
7    $F = \emptyset$ ;
8   for itemset  $I$  in  $C$  do
9     | Add  $I$  to  $F$  if support of  $I$  in  $\mathcal{D}$  is at least  $\theta n$  /* reads database */;
10  end
11  Output  $F$ ;
12 end

```

**Algorithm 1:** Apriori algorithm for frequent itemset mining

### 3 LSH-Apriori

The focus of this paper is to reduce the computation of processing all pairs of itemsets at each level in line 6 (which includes computing support by going through  $\mathcal{D}$ ). Suppose that level  $l$  outputs  $m_l$  frequent itemsets. We will treat the output of level  $l$  as a collection of  $m_l$  transaction vectors  $\mathcal{D}_l = \{I_1, \dots, I_{m_l}\}$ , each of length  $n$  and one for each frequent itemset of the  $l$ -th level. Our approach involves defining appropriate notions of similarity between itemsets (represented by vectors) in  $\mathcal{D}_l$  similar to the approach followed by Cohen *et al.*[4]. Let  $I_i, I_j$  be two vectors each of length  $n$ . Then, we use  $|I_i, I_j|$  to denote the number of bit positions where both the vectors have a 1.

**Definition 4.** *Given a parameter  $0 < \varepsilon < 1$ , we say that  $\{I_i, I_j\}$  is  $\theta$ -frequent (or similar) if  $|I_i, I_j| \geq \theta n$  and  $\{I_i, I_j\}$  is  $(1-\varepsilon)\theta$ -infrequent if  $|I_i, I_j| < (1-\varepsilon)\theta n$ . Furthermore, we say that  $I_j$  is similar to  $I_i$  if  $\{I_i, I_j\}$  is  $\theta$ -frequent.*

Let  $I_q$  be a frequent itemset at level  $l-1$ . Let  $\text{FI}(I_q, \theta)$  be the set of itemsets  $I_a$  such that  $\{I_q, I_a\}$  is  $\theta$ -frequent at level  $l$ . Our main contributions are a few randomized algorithms for identifying itemsets in  $\text{FI}(I_q, \theta)$  with high-probability.

**Definition 5** ( $\text{FI}(I_q, \theta, \varepsilon, \delta)$ ). *Given a  $\theta$ -frequent itemset  $I_q$  of size  $l-1$ , tolerance  $\varepsilon \in (0, 1)$  and error probability  $\delta$ ,  $\text{FI}(I_q, \theta, \varepsilon, \delta)$  is a set  $F'$  of itemsets of size  $l$ , such that with probability at least  $1 - \delta$ ,  $F'$  contains every  $I_a$  for which  $\{I_q, I_a\}$  is  $\theta$ -frequent.*

It is clear that  $\text{FI}(I_q, \theta) \subseteq \text{FI}(I_q, \theta, \varepsilon, \delta)$  with high probability. This motivated us to propose LSH-Apriori, a randomized version of Apriori, that takes  $\delta$  and  $\varepsilon$  as additional inputs and essentially replaces line 6 by LSH operations to combine every itemset  $I_q$  with only similar itemsets, unlike Apriori which combines all pairs of itemsets. This potentially creates a significantly smaller  $C$  without missing out too many frequent itemsets. The modifications to Apriori are presented in Algorithm 2 and the following lemma, immediate from Definition 5, establishes correctness of LSH-Apriori.

**Input:**  $\mathcal{D}_l = \{I_1, \dots, I_{m_l}\}$ ,  $\theta$ , (Additional) error probability  $\delta$ , tolerance  $\varepsilon$ ;  
**6a** (Pre-processing) Initialize hash tables and add all items  $I_a \in \mathcal{D}_l$ ;  
**6b** (Query) Compute  $\text{FI}(I_q, \theta, \varepsilon, \delta) \forall I_q \in \mathcal{D}_l$  by hashing  $I_q$  and checking collisions;  
**6c**  $C \leftarrow \{I_q \cup I_b \mid I_q \in \mathcal{D}_l, I_b \in \text{FI}(I_q, \theta, \varepsilon, \delta)\};$

**Algorithm 2:** LSH-Apriori level  $l+1$  (only modifications to Apriori line:6)

**Lemma 2** *Let  $I_q$  and  $I_a$  be two  $\theta$ -frequent compatible itemsets of size  $(l-1)$  such that the itemset  $J = I_q \cup I_a$  is also  $\theta$ -frequent. Then, with probability at least  $1 - \delta$ ,  $\text{FI}(I_q, \theta, \varepsilon, \delta)$  contains  $I_a$  (hence  $C$  contains  $J$ ).*

In the next section we describe three LSH-based randomized algorithms to compute  $\text{FI}(I_q, \theta, \varepsilon, \delta)$  for all  $\theta$ -frequent itemset  $I_q$  from the earlier level. The

input to these subroutines will be  $\mathcal{D}_l$ , the frequent itemsets from earlier level, and parameters  $\theta, \varepsilon, \delta$ . In the *pre-processing stage* at level  $l$ , the respective LSH is initialized and itemsets of  $\mathcal{D}_l$  are hashed; we specifically record the itemsets hashing to every bucket. LSH guarantees (*w.h.p.*) that pairs of similar items hash into the same bucket, and those that are not hash into different buckets. In the *query stage* we find all the itemsets that any  $I_q$  ought to be combined with by looking in the bucket in which  $I_q$  hashed, and then combining the compatible ones among them with  $I_q$  to form  $C$ . Rest of the processing happens à la Apriori.

The internal LSH subroutines may output false-positives – itemsets that are not  $\theta$ -frequent, but such itemsets are eventually filtered out in line 9 of Algorithm 1. Therefore, the output of LSH-Apriori does not contain any false positives. However, some frequent itemsets may be missing from its output (false negatives) with some probability depending on the parameter  $\delta$  as stated below in Theorem 3 (proof follows from the union bound and is given in the Appendix).

**Theorem 3 (Correctness).** *LSH-Apriori does not output any itemset which is not  $\theta$ -infrequent. If  $X$  is a  $\theta$ -frequent itemset of size  $l$ , then the probability that LSH-Apriori does not output  $X$  is at most  $\delta 2^l$ .*

The tolerance parameter  $\varepsilon$  can be used to balance the overhead from using hashing in LSH-Apriori with respect to its savings because of reading fewer transactions. Most LSH, including those that we will be using, behave somewhat like dimensionality reduction. As a result, the hashing operations do not operate on all bits of the vectors. Furthermore, the pre-condition of similarity for joining ensure that (*w.h.p.*) most infrequent itemsets can be detected before verifying them from  $\mathcal{D}$ . To formalize this, consider any level  $l$  with  $m_l$   $\theta$ -frequent itemsets  $\mathcal{D}_l$ . We will compare the computation done by LSH-Apriori at level  $l+1$  to what Apriori would have done at level  $l+1$  given the same frequent itemsets  $\mathcal{D}_l$ . Let  $c_{l+1}$  denote the number of candidates Apriori would have generated and  $m_{l+1}$  the number of frequent itemsets at this level (LSH-Apriori may generate fewer). **Overhead:** Let  $\tau(LSH)$  be the time required for hashing an itemset for a particular LSH and let  $\sigma(LSH)$  be the space needed for storing respective hash values. The extra overhead in terms of space will be simply  $m_l \sigma(LSH)$  in level  $l+1$ . With respect to overhead in running time, LSH-Apriori requires hashing each of the  $m_l$  itemsets twice, during pre-processing and during querying. Thus total time overhead in this level is  $\vartheta(LSH, l+1) = 2m_l \tau(LSH)$ .

**Savings:** Consider the itemsets in  $\mathcal{D}_l$  that are compatible with any  $I_q \in \mathcal{D}_l$ . Among them are those whose combination with  $I_q$  do not generate a  $\theta$ -frequent itemset for level  $l+1$ ; call them as *negative* itemsets and denote their number by  $r(I_q)$ . Apriori will have to read all  $n$  transactions of  $\sum_{I_q} r(I_q)$  itemsets in order to reject them. Some of these negative itemsets will be added to FI by LSH-Apriori – we will call them *false positives* and denote their count by  $FP(I_q)$ ; the rest those which correctly not added with  $I_q$  – let's call them as *true negatives* and denote their count by  $TN(I_q)$ . Clearly,  $r(I_q) = TN(I_q) + FP(I_q)$  and  $\sum_{I_q} r(I_q) = 2(c_{l+1} - m_{l+1})$ . Suppose  $\phi(LSH)$  denotes the number of transactions a particular LSH-Apriori reads for hashing any itemset; due to the dimensionality reduction

property of LSH,  $\phi(LSH)$  is always  $o(n)$ . Then, LSH-Apriori is able to reject all itemsets in  $TN$  by reading only  $\phi$  transactions for each of them; thus for itemset  $I_q$  in level  $l + 1$ , a particular LSH-Apriori reads  $(n - \phi(LSH)) \times TN(I_q)$  fewer transactions compared to a similar situation for Apriori. Therefore, total savings at level  $l + 1$  is  $\varsigma(LSH, l + 1) = (n - \phi(LSH)) \times \sum_{I_q} TN(I_q)$ .

In Section 4, we discuss this in more detail along with the respective LSH-Apriori algorithms.

## 4 FI via LSH

Our similarity measure  $|I_a, I_b|$  can also be seen as the inner product of the binary vectors  $I_a$  and  $I_b$ . However, it is not possible to get any LSH for such similarity measure because for example there can be three items  $I_a, I_b$  and  $I_c$  such that  $|I_a, I_b| \geq |I_c, I_c|$  which implies that  $\Pr(h(I_a) = h(I_b)) \geq \Pr(h(I_c) = h(I_c)) = 1$ , which is not possible. Noting the exact same problem, Shrivastava *et al.* introduced the concept of *asymmetric LSH* [11] in the context of binary inner product similarity. The essential idea is to use two different hash functions (for pre-processing and for querying) and they specifically proposed extending MinHashing by padding input vectors before hashing. We use the same pair of padding functions proposed by them for  $n$ -length binary vectors in a level  $l$ :  $P_{(n, \alpha_l)}$  for preprocessing and  $Q_{(n, \alpha_l)}$  for querying are defined as follows.

- In  $P(I)$  we append  $(\alpha_l n - |I|)$  many 1's followed by  $(\alpha_l n + |I|)$  many 0's.
- In  $Q(I)$  we append  $\alpha_l n$  many 0's, then  $(\alpha_l n - |I|)$  many 1's, then  $|I|$  0's.

Here,  $\alpha_l n$  (at LSH-Apriori level  $l$ ) will denote the maximum number of ones in any itemset in  $\mathcal{D}_l$ . Therefore, we always have  $(\alpha_l n - |I|) \geq 0$  in the padding functions. Furthermore, since the main loop of Apriori is not continued if no frequent itemset is generated at any level,  $(\alpha_l - \theta) > 0$  is also ensured at any level that Apriori is executing.

We use the above padding functions to reduce our problem of finding similar itemsets to finding nearby vectors under Hamming distance (using Hamming-based LSH in Subsection 4.1 and Covering LSH in Subsection 4.3) and under Jaccard similarity (using MinHashing in Subsection 4.2).

### 4.1 Hamming based LSH

In the following lemma (proof is given in appendix), we relate Hamming distance of two itemsets  $I_x$  and  $I_y$  with their  $|I_x, I_y|$ .

**Lemma 4** *For two itemsets  $I_x$  and  $I_y$ ,  $\text{Ham}(P(I_x), Q(I_y)) = 2(\alpha_l n - |I_x, I_y|)$ .*

Therefore, it is possible to use an LSH for Hamming distance to find similar itemsets. We use this technique in the following algorithm to compute  $\text{FI}(I_q, \theta, \varepsilon, \delta)$  for all itemset  $I_q$ . The algorithm contains an optimization over the generic LSH-Apriori pseudocode (Algorithm 2). There is no need to separately



execute lines:7–10 of Apriori; one can immediately set  $F \leftarrow C$  since LSH-Apriori computes support before populating FI.

**Input:**  $\mathcal{D}_l = \{I_1, \dots, I_{m_l}\}$ , query item  $I_q$ , threshold  $\theta$ , tolerance  $\varepsilon$ , error  $\delta$ .  
**Result:**  $\text{FI}_q = \text{FI}(I_q, \theta, \varepsilon, \delta)$  for every  $I_q \in \mathcal{D}_l$ .

**6a Preprocessing step:** Setup hash tables and add vectors in  $\mathcal{D}_l$ ;

- i Set  $\rho = \frac{\alpha_l - \theta}{\alpha_l - (1 - \varepsilon)\theta}$ ,  $k = \log\left(\frac{1 + 2\alpha_l}{(1 + 2(1 - \varepsilon)\theta)}\right) m_l$  and  $L = m_l^\rho \log\left(\frac{1}{\delta}\right)$ ;
- ii Select functions  $g_1, \dots, g_L$  u.a.r.;
- iii For every  $I_a \in \mathcal{D}_l$ , pad  $I_a$  using  $P()$  and then hash  $P(I_a)$  into buckets  $g_1(P(I_a)), \dots, g_L(P(I_a))$ ;

**6b Query step:** For every  $I_q \in \mathcal{D}_l$ , we do the following ;

- i  $S \leftarrow$  all  $I_q$ -compatible itemsets in all buckets  $g_i(Q(I_q))$ , for  $i = 1 \dots L$ ;
- ii **for**  $I_a \in S$  **do**
  - If  $|I_a, I_q| \geq \theta n$ , then add  $I_a$  to  $\text{FI}_q$  /\* reads database\*/;
  - (\*) If no itemset similar to  $I_q$  found within  $\frac{L}{\delta}$  tries, then break loop;

**end**

**Algorithm 3:** LSH-Apriori (only lines 6a,6b) using Hamming LSH

Correctness of this algorithm is straightforward. Also,  $\rho < 1$  and the space required and overhead of reading transactions is  $\theta(kLm_l) = o(m_l^2)$ . It can be further shown that  $\mathbb{E}[FP(I_q)] \leq L$  for  $I_q \in \mathcal{D}_l$  which can be used to prove that  $\mathbb{E}[\zeta] \geq (n - \phi)(2(c_{l+1} - m_{l+1}) - m_l L)$  where  $\phi = kL$ . Details of these calculations including complete proof of the next lemma is given in Appendix.

**Lemma 5** *Algorithm 3 correctly outputs  $\text{FI}(I_q, \theta, \varepsilon, \delta)$  for all  $I_q \in \mathcal{D}_l$ . Additional space required is  $o(m_l^2)$ , which is also the total time overhead. The expected savings can be bounded by  $\mathbb{E}[\zeta(l+1)] \geq (n - o(m_l))((c_{l+1} - 2m_{l+1}) + (c_{l+1} - o(m_l^2)))$ .*

Expected savings outweigh time overhead if  $n \gg m_l$ ,  $c_{l+1} = \theta(m_l^2)$  and  $c_{l+1} > 2m_{l+1}$ , i.e., in levels where the number of frequent itemsets generated are fewer compared to the number of transactions as well as to the number of candidates generated. The additional optimisation (\*) essentially increases the savings when all  $l + 1$ -extensions of  $I_q$  are  $(1 - \varepsilon)\theta$ -infrequent — this behaviour will be predominant in the last few levels. It is easy to show that in this case,  $FP(I_q) \leq \frac{L}{\delta}$  with probability at least  $1 - \delta$ ; this in turn implies that  $|S| \leq \frac{L}{\delta}$ . So, if we did not find any similar  $I_a$  within first  $\frac{L}{\delta}$  tries, then we can be sure, with reasonable probability, that there are no itemsets similar to  $I_q$ .

## 4.2 Min-hashing based LSH

Cohen et al. had given an LSH-based randomized algorithm for finding interesting itemsets without any requirement for high support [4]. We observed that their Minhashing-based technique [3] cannot be directly applied to the high-support version that we are interested in. The reason is roughly that Jaccard similarity and itemset similarity (w.r.t.  $\theta$ -frequent itemsets) are not monotonic



to each other. Therefore, we used padding to monotonically relate Jaccard similarity of two itemsets  $I_x$  and  $I_y$  with their  $|I_x, I_y|$  (proof is given in Appendix).

**Lemma 6** For two padded itemsets  $I_x$  and  $I_y$ ,  $\text{JS}(P(I_x), Q(I_y)) = \frac{|I_x, I_y|}{2\alpha_l n - |I_x, I_y|}$ .

Once padded, we follow similar steps (as [4]) to create a *similarity preserving summary*  $\hat{D}_l$  of  $\mathcal{D}_l$  such that the Jaccard similarity for any column pair in  $\mathcal{D}_l$  is approximately preserved in  $\hat{D}_l$ , and then explicitly compute  $\text{FI}(I_q, \theta, \varepsilon, \delta)$  from  $\hat{D}_l$ .  $\hat{D}_l$  is created by using  $\lambda$  independent minwise hashing functions (see Definition 3).  $\lambda$  should be carefully chosen since a higher value increases the accuracy of estimation, but at the cost of *large summary vectors* in  $\hat{D}_l$ . Let us define  $\hat{\text{JS}}(I_i, I_j)$  as the fraction of rows in the summary matrix in which min-wise entries of columns  $I_i$  and  $I_j$  are identical. Then by Theorem 1 of Cohen et al. [4], we can get a bound on the number of required hash functions:

**Theorem 7 (Theorem 1 of [4]).** Let  $0 < \epsilon, \delta < 1$  and  $\lambda \geq \frac{2}{\omega \epsilon^2} \log \frac{1}{\delta}$ . Then for all pairs of columns  $I_i$  and  $I_j$  following are true with probability at least  $1 - \delta$ :

- If  $\text{JS}(I_i, I_j) \geq s^* \geq \omega$ , then  $\hat{\text{JS}}(I_i, I_j) \geq (1 - \epsilon)s^*$ ,
- If  $\text{JS}(I_i, I_j) \leq \omega$ , then  $\hat{\text{JS}}(I_i, I_j) \leq (1 + \epsilon)\omega$ .

**Input:**  $\mathcal{D}_l$ , query item  $I_q$ , threshold  $\theta$ , tolerance  $\varepsilon$ , error  $\delta$   
**Result:**  $\text{FI}_q = \text{FI}(I_q, \theta, \varepsilon, \delta)$  for every  $I_q \in \mathcal{D}_l$ .

**6a Preprocessing step:** Prepare  $\hat{D}_l$  via MinHashing;

- i Set  $\omega = \frac{(1-\varepsilon)\theta}{2\alpha_l - (1-\varepsilon)\theta}$ ,  $\epsilon = \frac{\alpha_l \varepsilon}{\alpha_l + (\alpha_l - \theta)(1-\varepsilon)}$  and  $\lambda = \frac{2}{\omega \epsilon^2} \log \frac{1}{\delta}$ ;
- ii Choose  $\lambda$  many independent permutations (see Theorem 7);
- 4 iii For every  $I_a \in \mathcal{D}_l$ , pad  $I_a$  using  $P()$  and then hash  $P(I_a)$  using  $\lambda$  independent permutations;

**6b Query step:** For every  $I_q \in \mathcal{D}_l$ , we do the following ;

- i Hash  $Q(I_q)$  using  $\lambda$  independent permutations;
- ii **for compatible**  $I_a \in \mathcal{D}_l$  **do**
  - | If  $\hat{\text{JS}}(P(I_a), Q(I_q)) \geq \frac{(1-\varepsilon)\theta}{2\alpha_l - \theta}$  for some  $I_a$ , then add  $I_a$  to  $\text{FI}_q$ ;

**end**

**Algorithm 4:** LSH-Apriori (only lines 6a,6b) using Minhash LSH

**Lemma 8** Algorithm 4 correctly computes  $\text{FI}(I_q, \theta, \varepsilon, \delta)$  for all  $I_q \in \mathcal{D}_l$ . Additional space required is  $O(\lambda m_l)$ , and the total time overhead is  $O((n + \lambda)m_l)$ . The expected savings is given by  $\mathbb{E}[\zeta(l + 1)] \geq 2(1 - \delta)(n - \lambda)(c_{l+1} - m_{l+1})$ .

See Appendix for details of the above proof. Note that  $\lambda$  depends on  $\alpha_l$  but is independent of  $n$ . This method should be applied only when  $\lambda \ll n$ . And in that case, for levels with number of candidates much larger than the number of frequent itemsets discovered (i.e.,  $c_{l+1} \gg \{m_l, m_{l+1}\}$ ), time overhead would not appear significant compared to expected savings.

<sup>4</sup> This algorithm can be easily boosted to  $o(\lambda m_l)$  time by applying banding technique (see Section 4 of [4]) on the minhash table.

### 4.3 Covering LSH

Due to their probabilistic nature, the LSH-algorithms presented earlier have the limitation of producing false positives and more importantly, false negatives. Since the latter cannot be detected unlike the former, these algorithms may miss some frequent itemsets (see Theorem 3). In fact, once we miss some FI at a particular level, then all the FI which are “supersets” of that FI (in the subsequent levels) will be missed. Here we present another algorithm for the same purpose which overcomes this drawback. The main tool is a recent algorithm due to Pagh [9] which returns approximate nearest neighbors in the Hamming space. It is an improvement over the seminal LSH algorithm by Indyk and Motwani [7], also for Hamming distance. Pagh’s algorithm has a small overhead over the latter; to be precise, the query time bound of [9] differs by at most  $\ln(4)$  in the exponent in comparison with the time bound of [7]. However, its big advantage is that it generates no false negatives. Therefore, this LSH-Apriori version also does not miss any frequent itemset.

The LSH by Pagh is with respect to Hamming distance, so we first reduce our FI problem into the Hamming space by using the same padding given in Lemma 4. Then we use this LSH in the same manner as in Subsection 4.1. Pagh coined his hashing scheme as coveringLSH which broadly mean that given a threshold  $r$  and a tolerance  $c > 1$ , the hashing scheme guaranteed a collision for every pair of vectors that are within radius  $r$ . We will now briefly summarize coveringLSH for our requirement; refer to the paper [9] for full details.

Similar to HammingLSH, we use a family of Hamming projections as our hash functions:  $\mathcal{H}_A := \{x \mapsto x \wedge a \mid a \in \mathcal{A}\}$ , where  $\mathcal{A} \subseteq \{0, 1\}^{(1+2\alpha_l)n}$ . Now, given a query item  $I_q$ , the idea is to iterate through all hash functions  $h \in \mathcal{H}_A$ , and check if there is a collision  $h(P(I_x)) = h(Q(I_q))$  for  $I_x \in \mathcal{D}_l$ . We say that this scheme doesn’t produce false negative for the threshold  $2(\alpha_l - \theta)n$ , if at least one collision happens when there is an  $I_x \in \mathcal{D}_l$  when  $\text{Ham}(P(I_x), Q(I_q)) \leq 2(\alpha_l - \theta)n$ , and the scheme is efficient if the number of collision is not too many when  $\text{Ham}(P(I_x), Q(I_q)) > 2(\alpha_l - (1 - \varepsilon)\theta)n$  (proved in Theorem 3.1, 4.1 of [9]). To make sure that all pairs of vector within distance  $2(\alpha_l - \theta)n$  collide for some  $h$ , we need to make sure that some  $h$  map their “mismatching” bit positions (between  $P(I_x)$  and  $Q(I_q)$ ) to 0. We describe construction of hash functions next.

$n'$	$\theta'$	$t$	$c$	$\epsilon$	$\nu$
$(1+2\alpha_l)n$	$2(\alpha_l - \theta)n$	$\lceil \frac{\ln m_l}{2(\alpha_l - (1-\varepsilon)\theta)n} \rceil$	$\frac{\alpha_l - (1-\varepsilon)\theta}{\alpha_l - \theta}$	$\epsilon \in (0, 1) \text{ s.t. } \frac{\ln m_l}{2(\alpha_l - (1-\varepsilon)\theta)n} + \epsilon \in \mathbb{N}$	$\frac{t+\epsilon}{ct}$

*CoveringLSH:* The parameters relevant to LSH-Apriori are given above. Notice that after padding, dimension of each item is  $n'$ , threshold is  $\theta'$  (i.e., min-support is  $\theta'/n'$ ), and tolerance is  $c$ . We start by choosing a random function  $\varphi : \{1, \dots, n'\} \rightarrow \{0, 1\}^{t\theta'+1}$ , which maps bit positions of the padded itemsets to bit vectors of length  $t\theta' + 1$ . We define a family of bit vectors  $a(v) \in \{0, 1\}^{n'}$ , where  $a(v)_i = \langle \varphi(i), v \rangle$ , for  $i \in \{1, \dots, n'\}$ ,  $v \in \{0, 1\}^{t\theta'+1}$  and  $\langle m(i), v \rangle$  denotes the inner product over  $\mathbb{F}_2$ . We define our hash function family  $\mathcal{H}_A$  using all such vectors  $a(v)$  except  $a(\mathbf{0})$ :  $\mathcal{A} = \{a(v) \mid v \in \{0, 1\}^{t\theta'+1} / \{\mathbf{0}\}\}$ .

Pagh described how to construct  $\mathcal{A}' \subseteq \mathcal{A}$  [9, Corollary 4.1] such that  $\mathcal{H}_{\mathcal{A}'}$  has a very useful property of no false negatives and also ensuring very few false positives. We use  $\mathcal{H}_{\mathcal{A}'}$  for hashing using the same manner of Hamming projections as used in Subsection 4.1. Let  $\psi$  be the expected number of collisions between any itemset  $I_q$  and items in  $\mathcal{D}_l$  that are  $(1 - \varepsilon)\theta$ -infrequent with  $I_q$ . The following Theorem captures the essential property of coveringLSH that is relevant for LSH-Apriori, described in Algorithm 5. It also bounds the number of hash functions which controls the space and time overhead of LSH-Apriori. Proof of this theorem follows from Theorem 4.1 and Corollary 4.1 of [9].

**Theorem 9.** *For a randomly chosen  $\varphi$ , a hash family  $\mathcal{H}_{\mathcal{A}'}$  described above and distinct  $I_x, I_q \in \{0, 1\}^n$ :*

- *If  $\text{Ham}(P(I_x), Q(I_q)) \leq \theta'$ , then there exists  $h \in \mathcal{H}_{\mathcal{A}'}$  s.t.  $h(P(I_x)) = h(Q(I_q))$ ,*
- *Expected number of false positives is bounded by  $\mathbb{E}[\psi] < 2^{\theta' \epsilon + 1} m_l^{\frac{1}{c}}$ ,*
- *$|\mathcal{H}_{\mathcal{A}'}| < 2^{\theta' \epsilon + 1} m_l^{\frac{1}{c}}$ .*

**Input:**  $\mathcal{D}_l$ , query item  $I_q$ , threshold  $\theta$ , tolerance  $\varepsilon$ , error  $\delta$ .  
**Result:**  $\text{FI}_q = \text{FI}(I_q, \theta, \varepsilon, \delta)$  for every  $I_q \in \mathcal{D}_l$ .  
**6a Preprocessing step:** Setup hash tables according to  $\mathcal{H}_{\mathcal{A}'}$  and add items;  
   **i** For every  $I_a \in \mathcal{D}_l$ , hash  $P(I_a)$  using all  $h \in \mathcal{H}_{\mathcal{A}'}$ ;  
**6b Query step:** For every  $I_q \in \mathcal{D}_l$ , we do the following ;  
   **i**  $S \leftarrow$  all itemsets that collide with  $Q(I_q)$ ;  
   **ii for**  $I_a \in S$  **do**  
     If  $|I_a, I_q| \geq \theta n$ , then add  $I_a$  to  $\text{FI}_q$  /\* reads database\*/;  
     (\*) If no itemset similar to  $I_q$  found within  $\frac{\psi}{\delta}$  tries, break loop;  
   **end**

**Algorithm 5:** LSH-Apriori (only lines 6a,6b) using Covering LSH

**Lemma 10** *Algorithm 5 outputs all  $\theta$ -frequent itemsets and only  $\theta$ -frequent itemsets. Additional space required is  $O(m_l^{1+\nu})$ , which is also the total time overhead. The expected savings is given by  $\mathbb{E}[\zeta(l+1)] \geq 2 \left( n - \frac{\log m_l}{c} - 1 \right) ((c_{l+1} - m_{l+1}) - m_l^{1+\nu})$ .*

See Appendix for the proof. The (\*) line is an additional optimisation similar to what we did for HammingLSH 4.1; it efficiently recognizes those frequent itemsets of the earlier level none of whose extensions are frequent. The guarantee of not missing any valid itemset comes with a heavy price. Unlike the previous algorithms, the conditions under which expected savings beats overhead are quite stringent, namely,  $c_{l+1} \in \{\omega(m_l^2), \omega(m_{l+1}^2)\}$ ,  $\frac{2^n}{5} > m_l > 2^{n/2}$  and  $\epsilon < 0.25$  (since  $1 < c < 2$ , these bounds ensure that  $\nu < 1$  for later levels when  $\alpha_l \approx \theta$ ).

## 5 Conclusion

In this work, we designed randomized algorithms using locality-sensitive hashing (LSH) techniques which efficiently outputs almost all the frequent itemsets with

high probability at the cost of a little space which is required for creating hash tables. We showed that time overhead is usually small compared to the savings we get by using LSH.

Our work opens the possibilities for addressing a wide range of problems that employ on various versions of frequent itemset and sequential pattern mining problems, which potentially can efficiently be randomized using LSH techniques.

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## Appendix

**Theorem 3 (Correctness).** *LSH-Apriori does not output any itemset which is not  $\theta$ -infrequent. If  $X$  is a  $\theta$ -frequent itemset of size  $l$ , then the probability that LSH-Apriori does not output  $X$  is at most  $\delta 2^l$ .*

*Proof.* LSH-Apriori does not output  $X$ , whose size we denote by  $l$ , if at least one of these hold.

- Any 1 size subset of  $X$  is not generated by LSH-Apriori in level-1
- Any 2 size subset of  $X$  is not generated by LSH-Apriori in level-2
- $\vdots$
- Any  $l$  size subset of  $X$  (i.e.,  $X$  itself) is not generated in level- $l$

By Lemma 2,  $\delta$  is the probability that any particular frequent itemset is not generated at the right level, even though all its subsets were identified as frequent in earlier level. Since there are  $\binom{l}{k}$  subsets of  $X$  of size  $k$ , the required probability can be upper bounded using Union Bound to

$$\binom{l}{1}\delta + \binom{l}{2}\delta + \dots + \binom{l}{l}\delta \leq 2^l\delta.$$

To get the the necessary background, Lemma 13 provide bounds on the hashing parameters  $k, L$  for Hamming distance case. Their proof is adapted from [5, 7, 4]. We first require Lemma 11, 12 for the same.

**Lemma 11** *Let  $\{I_i, I_j\}$  be a pair of items s.t.  $\text{Ham}(I_i, I_j) \leq r$ , then the probability that  $I_i$  and  $I_j$  hash into at least one of the  $L$  bucket of size  $k$ , is at least  $1 - (1 - p_1^k)^L$ , where  $p_1 = 1 - \frac{r}{n}$ .*

*Proof.* Probability that  $I_i$  and  $I_j$  matches at some particular bit position  $\geq p_1$ . Now, probability that  $I_i$  and  $I_j$  matches at  $k$  positions in a bucket of size  $k \geq p_1^k$ . Probability that  $I_i$  and  $I_j$  don't matches at  $k$  positions in a bucket of size  $k \leq 1 - p_1^k$ . Probability that  $I_i$  and  $I_j$  don't matches at  $k$  positions in none of the  $L$  buckets  $\leq (1 - p_1^k)^L$ . Probability that  $I_i$  and  $I_j$  matches in at  $k$  positions positions in at least one of the  $L$  buckets  $\geq 1 - (1 - p_1^k)^L$ .

**Lemma 12** *Let  $\{I_i, I_j\}$  be a pair of items s.t.  $\text{Ham}(I_i, I_j) \geq (1 + \epsilon')r$ , then probability that  $\{I_i, I_j\}$  hash in a bucket of size  $k$ , is at most  $p_2^k$ , where  $p_2 = 1 - \frac{(1+\epsilon')r}{n}$ .*

*Proof.* Probability that  $I_i$  and  $I_j$  matches 1 at some particular bit position  $< p_2$ . Probability that  $I_i$  and  $I_j$  matches at  $k$  positions in a bucket of size  $k < p_2^k$ .

**Lemma 13** *Let  $\{I_i\}_{i=1}^m$  be a set of  $m$  vectors in  $R^n$ ,  $I_q$  be a given query vector, and  $I_{x^*}$  (with,  $1 \leq x^* \leq m$ ) s.t.  $\text{Ham}(I_{x^*}, I_q) \leq r$ . If we set our hashing parameters  $k = \log_{\frac{1}{p_2}} m$ , and  $L = m^\rho \log\left(\frac{1}{\delta}\right)$  (where,  $p_1 = 1 - \frac{r}{n}, p_2 = 1 - \frac{r(1+\epsilon')}{n}$ ,  $\rho = \frac{\log \frac{1}{p_1}}{\log \frac{1}{p_2}} \leq \frac{1}{1+\epsilon'}$ ), then the following two cases are true with probability  $> 1 - \delta$ :*

1. for some  $i \in \{1, \dots, L\}$ ,  $g_i(I_{x^*}) = g_i(I_q)$ ; and
2. total number of collisions with  $I_{x'}$  s.t.  $\text{Ham}(I_{x'}, I_q) > (1 + \epsilon')r$  is at most  $\frac{L}{\delta}$ .

*Proof.* Consider the first case, by Lemma 11, we have the following:

$$\Pr[\exists i : g_i(I_{x^*}) = g_i(I_q)] \geq 1 - (1 - p_1^k)^L.$$

If we choose  $k = \log_{\frac{1}{p_2}} m$ , we get  $p_1^k = p_1^{\log_{\frac{1}{p_2}} m} = m^{-\frac{\log \frac{1}{p_1}}{\log \frac{1}{p_2}}}$ . Let us denote  $\rho = \frac{\log \frac{1}{p_1}}{\log \frac{1}{p_2}}$ . Then,  $\Pr[\exists i : g_i(I_{x^*}) = g_i(I_q)] \geq 1 - (1 - m^{-\rho})^L$ . Now, if we set  $L = m^\rho \log(\frac{1}{\delta})$ , then the required probability is  $1 - (1 - m^{-\rho})^{m^\rho \log(\frac{1}{\delta})} \geq 1 - \frac{1}{\delta} > 1 - \delta$ .

Now, let us consider the case 2. Let  $I_{x'}$  be an item such that  $\text{Ham}(I_q, I_{x'}) > r(1 + \epsilon')$ . Then by Lemma 12, we have the following:

$$\Pr[g_i(I_q) = g_i(I_{x'})] \leq p_2^k = p_2^{\log_{\frac{1}{p_2}} m} = \frac{1}{m} \quad (\text{as we choosed } k = \log_{\frac{1}{p_2}} m).$$

Thus, the expected number of collisions for a particular  $i$  is at most 1, and the expected total number of collisions is at most  $L$  (by linearity of expectation). Now, by Markov's inequality  $\Pr[\text{Number of } I_{x'} \text{ which are colliding with } I_q > \frac{L}{\delta}] < \frac{L}{\delta} = \delta$ . Further,  $\rho = \frac{\log \frac{1}{p_1}}{\log \frac{1}{p_2}} = \frac{n-r}{n-(1+\epsilon')r} \leq \frac{1}{1+\epsilon'}$  (after simplification).

**Lemma 4** For two itemsets  $I_x$  and  $I_y$ ,  $\text{Ham}(P(I_x), Q(I_y)) = 2(\alpha_l n - |I_x, I_y|)$ .

*Proof.* It is easy to verify that with this mapping  $|P(I_x), Q(I_y)| = |I_x, I_y|$ . Let  $\text{Ham}(I_x, I_y)$  denote the hamming distance between items  $I_x$  and  $I_y$ . Then,  $\text{Ham}(P(I_x), Q(I_y)) = |P(I_x)| + |Q(I_y)| - 2|P(I_x), Q(I_y)| = \alpha_l n - |I_x| + |I_x| + \alpha_l n - |I_y| + |I_y| - 2|I_x, I_y| = 2(\alpha_l n - |I_x, I_y|)$ .

**Lemma 5** Algorithm 3 correctly outputs  $\text{FI}(I_q, \theta, \epsilon, \delta)$  for all  $I_q \in \mathcal{D}_l$ . Additional space required is  $o(m_l^2)$ , which is also the total time overhead. The expected savings can be bounded by  $\mathbb{E}[\zeta(l+1)] \geq (n - o(m_l))((c_{l+1} - 2m_{l+1}) + (c_{l+1} - o(m_l^2)))$ .

*Proof.* First, we show that for any query item  $I_q$ , Algorithm 3 correctly outputs  $\text{FI}(I_q, \theta, \epsilon, \delta)$  for any query  $I_q \in \mathcal{D}_l$ . Now, if there is an item  $I_{x^*}$  such that  $|I_{x^*}, I_q| \geq \theta n$ , then  $\text{Ham}(P(I_{x^*}), Q(I_q)) \leq 2(\alpha_l - \theta)n$  (by Lemma 4). Let  $p_1$  be the probability that  $P(I_{x^*})$  and  $Q(I_q)$  matches at some particular bit position, then  $p_1 \geq 1 - \frac{2(\alpha_l n - \theta n)}{n + 2\alpha_l n} = \frac{1+2\theta}{1+2\alpha_l}$ . Similarly, if there is an item  $I_{x'}$  such that  $|I_{x'}, I_q| \leq (1 - \epsilon)\theta n$ , then  $\text{Ham}(P(I_{x'}), Q(I_q)) \geq 2(\alpha_l - (1 - \epsilon)\theta)n$ . Let  $p_2$  be the probability that  $P(I_{x'})$  and  $Q(I_q)$  matches at some particular bit position, then  $p_2 \leq 1 - \frac{2(\alpha_l - (1 - \epsilon)\theta)n}{(1 + 2\alpha_l)n} = \frac{1+2(1-\epsilon)\theta}{1+2\alpha_l}$ .

Now, as we have set  $k = \log_{\frac{1}{p_2}} m = \log_{\left(\frac{1+2\alpha_l}{1+2(1-\epsilon)\theta}\right)} m_l$ ; and  $L = m_l^\rho \log \frac{1}{\delta}$ , where  $\rho =$

$\frac{\log \frac{1}{p_1}}{\log \frac{1}{p_2}} = \frac{\log \frac{1+2\theta}{1+2\alpha_l}}{\log \frac{1+2(1-\epsilon)\theta}{1+2\alpha_l}} = \frac{\alpha_l - \theta}{\alpha_l - (1-\epsilon)\theta}$  (after simplification); the proof easily follows from Lemma 13.

The space required for hashing an itemset  $I_q$  is  $\sigma = O(kL) = \tilde{O}(m_l^\rho) = m_l^{o(1)}$ , where  $\rho = \frac{\alpha_l - \theta}{\alpha_l - (1-\varepsilon)\theta} = o(1)$ . Time  $\tau$  required for hashing  $I_i$  is also  $O(kL) = \tilde{O}(m_l^\rho) = m_l^{o(1)}$ . Thus, total time and space overhead is  $m_l^{1+o(1)}$ , which immediately proves the required space and time overhead.

The number of bits of any vector required by this LSH-function is  $\phi = O(kL) = m_l^{o(1)}$ . We know that  $TN$  and  $FP$  for an itemset  $I_q \in \mathcal{D}_l$  are related by  $TN(I_q) = r(I_q) - FP(I_q)$ , and  $\sum_{I_q} r(I_q) = 2(c_{l+1} - m_{l+1})$ . So,  $\mathbb{E}[\sum TN(I_q)] = \sum r(I_q) - \mathbb{E}[\sum FP(I_q)]$ . From Lemma 13,  $\mathbb{E}[FP(I_q)] \leq L$ . Combining these facts, we get  $\mathbb{E}[\sum TN(I_q)] \geq 2(c_{l+1} - m_{l+1}) - m_l L = 2(c_{l+1} - m_{l+1}) - m_l m_l^\rho = 2(c_{l+1} - m_{l+1}) - m_l^{1+o(1)}$ . Now, using the formula for expected savings from Section 3,

$$\begin{aligned} \mathbb{E}[\zeta(l+1)] &\geq (n - O(kL)) \left( 2(c_{l+1} - m_{l+1}) - m_l^{1+o(1)} \right) \\ &= \left( n - m_l^{o(1)} \right) (2(c_{l+1} - m_{l+1}) - m_l^{1+o(1)}) \\ &\geq (n - o(m_l)) ((c_{l+1} - 2m_{l+1}) + (c_{l+1} - o(m_l^2))). \end{aligned}$$

**Lemma 6** For two padded itemsets  $I_x$  and  $I_y$ ,  $\text{JS}(P(I_x), Q(I_y)) = \frac{|I_x, I_y|}{2\alpha_l n - |I_x, I_y|}$ .

*Proof.* The Jaccard Similarity between items  $P(I_x)$  and  $Q(I_y)$  is as follows:

$$\begin{aligned} \text{JS}(P(I_x), Q(I_y)) &= \frac{|P(I_x) \cap Q(I_y)|}{|P(I_x) \cup Q(I_y)|} = \frac{|P(I_x), Q(I_y)|}{|P(I_x)| + |Q(I_y)| - |P(I_x), Q(I_y)|} \\ &= \frac{|I_x, I_y|}{\alpha_l n - |I_x| + |I_x| + \alpha_l n - |I_y| + |I_y| - |I_x, I_y|} = \frac{|I_x, I_y|}{2\alpha_l n - |I_x, I_y|}. \end{aligned}$$

**Lemma 8** Algorithm 4 correctly computes  $\text{FI}(I_q, \theta, \varepsilon, \delta)$  for all  $I_q \in \mathcal{D}_l$ . Additional space required is  $O(\lambda m_l)$ , and the total time overhead is  $O((n + \lambda)m_l)$ . The expected savings is given by  $\mathbb{E}[\zeta(l+1)] \geq 2(1 - \delta)(n - \lambda)(c_{l+1} - m_{l+1})$ .

*Proof.* Now, if there is an item  $I_{x^*}$  such that  $|I_{x^*}, I_q| \geq \theta n$ , then  $\text{JS}(P(I_{x^*}), Q(I_q)) \geq \frac{\theta n}{2\alpha_l n - \theta n}$  (by Lemma 6). As we set  $\omega = \frac{(1-\varepsilon)\theta}{2\alpha_l - (1-\varepsilon)\theta}$  in Algorithm 4, then by Theorem 7 we have  $\hat{\text{JS}}(P(I_{x^*}), Q(I_q)) \geq \frac{(1-\varepsilon)\theta n}{(2\alpha_l - \theta)n}$ , with probability at least  $1 - \delta$ .

Similarly, if there is an item  $I_{x'}$  such that  $|I_{x'}, I_q| < (1 - \varepsilon)\theta n$ , then  $\text{JS}(P(I_{x'}), Q(I_q)) < \frac{(1-\varepsilon)\theta n}{2(\alpha_l - (1-\varepsilon)\theta)n} = \frac{(1-\varepsilon)\theta}{2\alpha_l - (1-\varepsilon)\theta}$  (by Lemma 6). Then by Theorem 7, we have  $\hat{\text{JS}}(P(I_{x'}), Q(I_q)) < \frac{(1+\varepsilon)(1-\varepsilon)\theta}{2\alpha_l - (1-\varepsilon)\theta}$ , with probability at least  $1 - \delta$ .

We need to set Minhash parameter  $\epsilon$  such that  $\hat{\text{JS}}(P(I_a), Q(I_q)) \geq \frac{(1-\epsilon)\theta}{2\alpha_l - \theta} \geq \frac{(1+\epsilon)(1-\varepsilon)\theta}{2\alpha_l - (1-\varepsilon)\theta}$ . This gives,  $\epsilon < \frac{\alpha_l \varepsilon}{\alpha_l + (\alpha_l - \theta)(1-\varepsilon)}$ , which ensures  $|I_q, I_a| \geq (1 - \varepsilon)\theta n$  with probability at least  $1 - \delta$ .

The space required for hashing an itemset  $I_i$  (for  $1 \leq i \leq m_l$ ) is  $\sigma = O(\lambda)$ . Where,  $\lambda \geq \frac{2}{\omega \epsilon^2} \log \frac{1}{\delta}$  and  $\epsilon = \frac{\alpha_l \varepsilon}{\alpha_l + (\alpha_l - \theta)(1-\varepsilon)}$ . Total space required for storing hash table is  $O(m_l \lambda)$ . Creating hash table require one pass over  $\mathcal{D}_l$ , then pre-processing time overhead is  $O(n m_l)$ . We perform query on the hash table, query time overhead is  $O(\lambda m_l)$ . Thus, total time overhead  $\vartheta(l+1) = O((n + \lambda)m_l)$ .

Now, if an itemset  $I_a$  is infrequent with  $I_q$ , then  $\Pr[I_a \text{ is not reported}] \geq 1 - \delta$ . As there are  $(c_{l+1} - m_{l+1})$  number of infrequent itemsets at level  $l+1$ , then,



expected number of infrequent items that are not reported  $\mathbb{E}[TN] \geq (1-\delta)(c_{l+1} - m_{l+1})$ . Therefore,  $\mathbb{E}[\varsigma(l+1)] = (n-\lambda)\mathbb{E}[TN] \geq (n-\lambda)(1-\delta)(c_{l+1} - m_{l+1})$ .

**Lemma 10** *Algorithm 5 outputs all  $\theta$ -frequent itemsets and only  $\theta$ -frequent itemsets. Additional space required is  $O(m_l^{1+\nu})$ , which is also the total time overhead. The expected savings is given by  $\mathbb{E}[\varsigma(l+1)] \geq 2\left(n - \frac{\log m_l}{c} - 1\right)((c_{l+1} - m_{l+1}) - m_l^{1+\nu})$ .*

*Proof.* By Theorem 9, and our choice of hash function  $\mathcal{H}_{A'}$ , any pair of similar itemset will surely collide by our hash function, and will not get missed by the algorithm. Moreover, *false positives* will be filter out by the algorithm in Line 6b of Algorithm 5. Thus, our algorithm outputs all  $\theta$ -frequent itemsets and only  $\theta$ -frequent itemsets.

The space required for hashing an itemset  $I_q$  is  $\sigma = |\mathcal{H}_{A'}|$ . Total space required for creating hash table is  $O(m_l|\mathcal{H}_{A'}|) = O\left(m_l 2^{\theta'\epsilon+1} m_l^{\frac{1}{c}}\right) = O\left(m_l m_l^{\frac{\epsilon}{ct}} m_l^{\frac{1}{c}}\right) = O\left(m_l^{1+\frac{t+\epsilon}{ct}}\right) = O(m_l^{1+\nu})$ . Time  $\tau$  require for hashing  $I_i$  is also  $|\mathcal{H}_{A'}|$ . Thus, total time overhead required (including both preprocessing and querying) is  $\vartheta(l+1) = O(m_l|\mathcal{H}_{A'}|) = O(m_l^{1+\nu})$ , which proves the required space and time overhead.

The number of bits of any item required by our hash function is  $\phi = \frac{\log m_l}{c} + 1$ . We know that  $TN$  and  $FP$  for an itemset  $I_q \in \mathcal{D}_l$  are related by  $TN(I_q) = r(I_q) - FP(I_q)$ , and  $\sum_{I_q} r(I_q) = 2(c_{l+1} - m_{l+1})$ . So,  $\mathbb{E}[\sum TN(I_q)] = \sum r(I_q) - \mathbb{E}[\sum FP(I_q)]$ . From Theorem 9,  $\mathbb{E}[FP(I_q)] \leq \psi$ . Then, we get  $\mathbb{E}[\sum TN(I_q)] \geq 2(c_{l+1} - m_{l+1}) - m_l\psi$ .

As expected savings is  $\varsigma(LSH, l+1) = (n - \phi) \times \sum_{I_q} TN(I_q)$ . We have,

$$\begin{aligned} \mathbb{E}[\varsigma(l+1)] &\geq \left(n - \frac{\log m_l}{c} - 1\right) (2(c_{l+1} - m_{l+1}) - m_l\psi) \\ &\geq \left(n - \frac{\log m_l}{c} - 1\right) \left(2(c_{l+1} - m_{l+1}) - m_l 2^{\theta'\epsilon+1} m_l^{\frac{1}{c}}\right) \\ &\geq 2\left(n - \frac{\log m_l}{c} - 1\right) \left(2(c_{l+1} - m_{l+1}) - m_l^{1+\frac{t+\epsilon}{ct}}\right) \\ &= 2\left(n - \frac{\log m_l}{c} - 1\right) (2(c_{l+1} - m_{l+1}) - (m_l^{1+\nu})). \end{aligned}$$

We end with a quick proof that the following are sufficient to ensure overhead is less than expected savings.  $c_{l+1} \in \{\omega(m_l^2), \omega(m_{l+1}^2)\}$ ,  $\frac{2^n}{5} > m_l > 2^{n/2}$ ,  $\epsilon < 0.25$

and  $\alpha \approx \theta$ . Note that,  $2^{cn} > 2^n$  and  $5m_l > 2^{c \cdot c^{1/d}} m_l$  for any  $d > 0$ . These imply,

$$\begin{aligned} nc &> \log(m_l) + c \cdot c^{1/d} \\ n - c^{1/d} &> \frac{\log(m_l)}{c} \\ n - 1 &> \frac{\log(m_l)}{c} \quad \text{since, } c^{1/d} > 1 \end{aligned}$$

Furthermore, our conditions imply that  $4\epsilon(\alpha_l - \theta)c < c - 1$ . This implies,

$$\begin{aligned} \frac{2\epsilon(\alpha_l - \theta)c}{c - 1} &< 1/2 \\ n - 2 &> n/2 > \frac{2\epsilon(\alpha_l - \theta)cn}{c - 1} \\ \log(m) &> \frac{2\epsilon(\alpha_l - \theta)cn}{c - 1} \\ 1 &= \frac{1}{c} + \frac{c - 1}{c} > \frac{1}{c} + \frac{2\epsilon(\alpha_l - \theta)n}{\log(m)} = \frac{1}{c} + \frac{\epsilon}{ct} = \nu \end{aligned}$$